Quantum Computation- Physical Realization

System	τ_Q	τ_{op}	$n_{op} = \lambda^{-1}$
Nuclear spin	$10^{-2} - 10^{8}$	$10^{-3} - 10^{-6}$	$10^{5} - 10^{14}$
Electron spin	10^{-3}	10^{-7}	10^{4}
Ion trap (In^+)	10^{-1}	10^{-14}	10^{13}
Electron – Au	10^{-8}	10^{-14}	10^{6}
Electron – GaAs	10^{-10}	10^{-13}	10^{3}
Quantum dot	10^{-6}	10^{-9}	10^{3}
Optical cavity	10^{-5}	10^{-14}	10^{9}
Microwave cavity	10^{0}	10^{-4}	10^{4}

- Crude estimates for decoherence times τ_Q (seconds), operation times τ_{op} (seconds), and maximum number of operations $n_{op} = \lambda^{-1} = \tau_Q/\tau_{op}$ for various candidate physical realizations of interacting systems of quantum bits.

Quantum Computation- Physical Realization- Conditions for quantum computation

- Conditions for quantum computation
  - 1. Robustly represent quantum information
  - 2. Perform a universal family of unitary transformations
  - 3. Prepare a fiducial (fixed basis of reference) initial state
  - 4. Measure the output result

Quantum Computation- Physical Realization- Conditions for quantum computation

- Representation of quantum information
  - Quantum computation is based on transformation of quantum states.
  - Quantum bits are two-level quantum systems, and as the simplest elementary building blocks for a quantum computer, they provide a convenient labeling for pairs of states and their physical realizations.
  - For single qubits, the figure of merit is the minimum lifetime of arbitrary superposition states

Quantum Computation- Physical Realization- Conditions for quantum computation

- Square wells and qubits
  - A prototypical quantum system is known as the 'square well,' which is a particle in a one-dimensional box, behaving according to Schrodinger's equation
  - The Hamiltonian for this system is
  - $H = p^2/2m+V(x)$, where $V(x) = 0$ for $0 < x < L$, and $V(x) = \infty$ otherwise

Quantum Computation- Physical Realization- Conditions for quantum computation

- The energy eigenstates, expressed as wavefunctions in the position basis, are

$$|\psi_n\rangle = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi}{L}x\right)$$

- where n is an integer, and

$$|\psi_n(t)\rangle = e^{-iE_n t}|\psi_n\rangle \qquad E_n = n^2\pi^2 m/2L^2$$

Quantum Computation- Physical Realization- Conditions for quantum computation

- These states have a discrete spectrum.
- In particular, suppose that we arrange matters such that only the two lowest energy levels need be considered in an experiment.
- We define an arbitrary wavefunction of interest as $|\psi\rangle = a\,|\psi_1\rangle + b\,|\psi_2\rangle$. Since

$$|\psi(t)\rangle = e^{-i(E_1+E_2)/2t}\left[ae^{-i\omega t}|\psi_1\rangle + be^{i\omega t}|\psi_2\rangle\right]$$

Quantum Computation- Physical Realization- Conditions for quantum computation

- where $\omega = (E_1 - E_2)/2$, we can just forget about everything except a and b, and write our state abstractly as the two-component vector

$$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$$

- This two-level system represents a qubit!
- Does our two-level system transform like a qubit?

Quantum Computation- Physical Realization-

Conditions for quantum computation

- Under time evolution, this qubit evolves under the effective Hamiltonian H = ħωZ, which can be disregarded by moving into the rotating frame.
- To perform operations to this qubit, we perturb *H.*
- Consider the effect of adding the additional term to *V (x).*
- $$\delta V(x) = -V_0(t)\frac{9\pi^2}{16L}\left(\frac{x}{L}-\frac{1}{2}\right)$$

Quantum Computation- Physical Realization- Conditions for quantum computation

- Measurement of output result
  - Let us think of measurement as a process of coupling one or more qubits to a classical system such that after some interval of time, the state of the qubits is indicated by the state of the classical system.
  - For example, a qubit state $a|0\rangle + b|1\rangle$, represented by the ground and excited states of a two-level atom, might be measured by pumping the excited state and looking for fluorescence.

Quantum Computation- Physical Realization- Conditions for quantum computation

- If an electrometer indicates that fluorescence had been detected by a photomultiplier tube, then the qubit would collapse into the $|1>$ state; this would happen with probability $|b|^2$.
- Otherwise, the electrometer would detect no charge, and the qubit would collapse into the $|0>$ state.

Quantum Computation- Physical Realization- Conditions for quantum computation

- An important characteristic of the measurement process for quantum computation is the wavefunction collapse which describes what happens when a projective measurement is performed
- Many difficulties with measurement can be imagined; for example, inefficient photon counters and amplifier thermal noise can reduce the information obtained about measured qubit states in the scheme just described.
- A good figure of merit for measurement capability is the signal to noise ratio (SNR). This accounts for measurement inefficiency as well as inherent signal strength available from coupling a measurement apparatus to the quantum system.

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- Physical apparatus
  - An example of a simple harmonic oscillator is a particle in a parabolic potential well, $V(x) = m\omega^2 x^2/2$.
  - In the classical world, this could be a mass on a spring, which oscillates back and forth as energy is transferred between the potential energy of the spring and the kinetic energy of the mass.
  - It could also be a resonant electrical circuit, where the energy sloshes back and forth between the inductor and the capacitor. In these systems, the total energy of the system is a continuous parameter.

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- The set of discrete energy eigenstates of a simple harmonic oscillator can be labeled as $|n\rangle$, where $n = 0, 1, \ldots, \infty$. The relationship to quantum computation comes by taking a finite subset of these states to represent qubits.
- These qubits will have lifetimes determined by physical parameters such as the cavity quality factor Q, which can be made very large by increasing the reflectivity of the cavity walls.

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- The Hamiltonian
  - The Hamiltonian for a particle in a one-dimensional parabolic potential is

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

  - where p is the particle momentum operator, m is the mass, x is the position operator, and ω is related to the potential depth.

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- Recall that x and p are operators in this expression which can be rewritten as

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

- where $a^\dagger$ and a are creation and annihilation operators, defined as

$$a = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x + ip\right)$$

- $$a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x - ip\right)$$

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- The zero point energy ħω/2 contributes an unobservable overall phase factor, which can be disregarded for our present purpose.
- The eigenstates |n> of H, where n = 0, 1, . . ., have the properties

$$
\begin{aligned}
a^\dagger a|n\rangle &= n|n\rangle \\
a^\dagger|n\rangle &= \sqrt{n+1}\,|n+1\rangle \\
a|n\rangle &= \sqrt{n}\,|n-1\rangle\,.
\end{aligned}
$$

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- Later, we will find it convenient to express interactions with a simple harmonic oscillator by introducing additional terms involving a and $a^\dagger$, and interactions between oscillators with terms such as $a^\dagger_1 a_2 + a_1 a^\dagger_2$. For now, however, we confine our attention to a single oscillator.

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- Exercise: Using the fact that x and p do not commute, and that in fact $[x, p] = i\hbar$, explicitly show that $a^\dagger a = H/\hbar\omega - 1/2$.
- Exercise: Given that $[x, p] = i\hbar$, compute $[a, a^\dagger]$.
- Exercise: Show that

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$$

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- Time evolution of the eigenstates is given by solving the Schodinger equation from which we find that the state

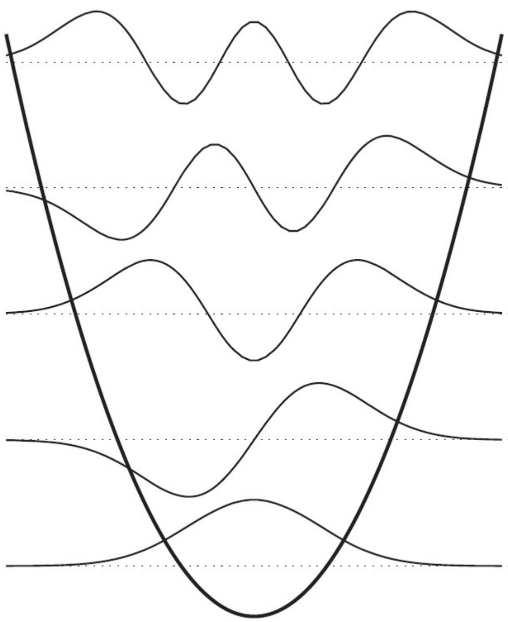
$$|\psi(0)\rangle = \sum_n c_n(0)|n\rangle$$

- evolves in time to become

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle = \sum_n c_n e^{-in\omega t}|n\rangle$$

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- We will assume for the purpose of discussion that an arbitrary state can be perfectly prepared, and that the state of the system can be projectively measured



$$\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \psi_n(x) = E\psi_n(x)$$

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- Quantum computation
  - Suppose we want to perform quantum computation with the single simple harmonic oscillator described above. What can be done?
  - The most natural choice for representation of qubits are the energy eigenstates $|n>$.

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- This choice allows us to perform a controlled-NOT gate in the following way. Recall that this transformation performs the mapping

$$
\begin{aligned}
|00\rangle_L &\rightarrow |00\rangle_L \\
|01\rangle_L &\rightarrow |01\rangle_L \\
|10\rangle_L &\rightarrow |11\rangle_L \\
|11\rangle_L &\rightarrow |10\rangle_L
\end{aligned}
$$

- on two qubit states (here, the subscript L is used to clearly distinguish 'logical' states in contrast to the harmonic oscillator basis states).

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- Let us encode these two qubits using the mapping

$$
\begin{aligned}
|00\rangle_L &= |0\rangle \\
|01\rangle_L &= |2\rangle \\
|10\rangle_L &= (|4\rangle + |1\rangle)/\sqrt{2} \\
|11\rangle_L &= (|4\rangle - |1\rangle)/\sqrt{2}
\end{aligned}
$$

- Now suppose that at $t = 0$ the system is started in a state spanned by these basis states, and we simply evolve the system forward to time $t = \pi/\hbar\omega$.

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- This causes the energy eigenstates to undergo the transformation $|n\rangle \rightarrow \exp(-i\pi a^{\dagger}a)|n\rangle = (-1)^n|n\rangle$, such that $|0\rangle$, $|2\rangle$, and $|4\rangle$ stay unchanged, but $|1\rangle \rightarrow -|1\rangle$.
- As a result, we obtain the desired controlled-NOT gate transformation.
- This suggests that perhaps one might be able to realize an entire quantum computer in a single simple harmonic oscillator!

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- Drawbacks
  - One will not always know the eigenvalue spectrum of the unitary operator for a certain quantum computation, even though one may know how to construct the operator from elementary gates. In fact, for most problems addressed by quantum algorithms, knowledge of the eigenvalue spectrum is tantamount to knowledge of the solution!

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- Another obvious problem is that the technique used above does not allow one computation to be cascaded with another, because in general, cascading two unitary transforms results in a new transform with unrelated eigenvalues.

- It neglects the principle of digital representation of information. Quantum computation builds upon *digital* computation, not analog computation.

Quantum Computation- Physical Realization- Harmonic oscillator quantum computer

- Summary
  - Qubit representation: Energy levels |0>, |1>, . . ., $|2^n>$ of a single quantum oscillator give n qubits.
  - Unitary evolution: Arbitrary transforms U are realized by matching their eigenvalue spectrums to that given by the Hamiltonian $H = a^\dagger a$.
  - Initial state preparation: Not considered.
  - Readout: Not considered.
  - Drawbacks: Not a digital representation! Also, matching eigenvalues to realize transformations is not feasible for arbitrary U, which generally have unknown eigenvalues.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Photons:
  - An attractive physical system for representing a quantum bit is the optical photon
  - Photons are charge-less particles, and do not interact very strongly with each other, or even with most matter.
  - They can be guided along long distances with low loss in optical fibers, delayed efficiently using phase shifters, and combined easily using beam-splitters.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Photons exhibit signature quantum phenomena, such as the interference produced in two-slit experiments.

- In principle, photons *can* be made to interact with each other, using nonlinear optical media which mediate interactions.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Physical apparatus
  - The energy in an electromagnetic cavity is quantized in units of $\hbar\omega$.
  - Each such quantum is called a photon.
  - It is possible for a cavity to contain a superposition of zero or one photon, a state which could be expressed as a qubit $c_0|0\rangle + c_1|1\rangle$, but we shall do something different.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Let us consider two cavities, whose total energy is $\hbar\omega$, and take the two states of a qubit as being whether the photon is in one cavity ($|01>$) or the other ($|10>$).

- The physical state of a superposition would thus be written as $c_0|01> + c_1|10>$; we shall call this the dual-rail representation.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Note that we shall focus on single photons traveling as a wave-packet through free space, rather than inside a cavity; one can imagine this as having a cavity moving along with the wave-packet.

- Each cavity in our qubit state will thus correspond to a different spatial mode.

Quantum Computation- Physical Realization- Optical photon quantum computer

- One scheme for generating single photons in the laboratory is by attenuating the output of a laser.

- A laser outputs a state known as a coherent state, $|\alpha\rangle$, defined as

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Quantum Computation- Physical Realization- Optical photon quantum computer

– where |n> is an n-photon energy eigen-state.

– This state, which has been the subject of thorough study in the field of quantum optics, has many beautiful properties which we shall not describe here.

– It suffices to understand just that coherent states are naturally radiated from driven oscillators such as a laser when pumped high above its lasing threshold.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Note that the mean energy is $\langle\alpha|n|\alpha\rangle = |\alpha|^2$.
- When attenuated, a coherent state just becomes a weaker coherent state, and a weak coherent state can be made to have just one photon, with high probability.

- Exercise: (Eigen-states of photon annihilation) Prove that a coherent state is an eigen-state of the photon annihilation operator, that is, show $a|\alpha\rangle = \lambda|\alpha\rangle$ for some constant λ.

Quantum Computation- Physical Realization- Optical photon quantum computer

- For example, for $\alpha = \sqrt{0.1}$
- We obtain the state

$$\sqrt{0.90}\,|0\rangle + \sqrt{0.09}\,|1\rangle + \sqrt{0.002}\,|2\rangle + \cdots.$$

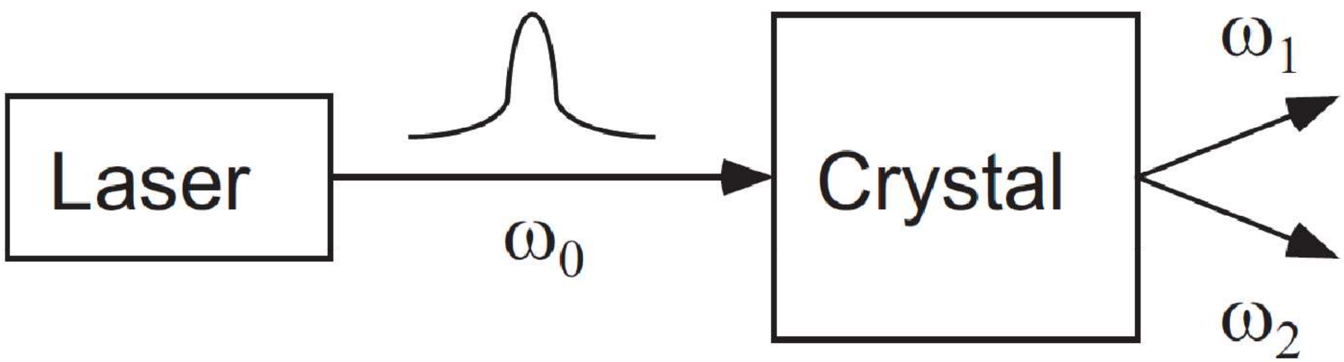
- Thus if light ever makes it through the attenuator, one knows it is a single photon with probability better than 95%; the failure probability is thus 5%.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Note also that 90% of the time, no photons come through at all; this source thus has a rate of 0.1 photons per unit time.

- Finally, this source does not indicate (by means of some classical readout) when a photon has been output or not; two of these sources cannot be synchronized.

Quantum Computation- Physical Realization- Optical photon quantum computer



Parametric down-conversion scheme for generation of single photons.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Better synchronicity can be achieved using parametric down-conversion. This involves sending photons of frequency ω_0 into a nonlinear optical medium such as KH_2PO_4 to generate photon pairs at frequencies $\omega_1 + \omega_2 = \omega_0$.

- Momentum is also conserved, such that $k_1+k_2 = k_3$, so that when a single ω_2 photon is (destructively) detected, then a single ω_1 photon is known to exist

Quantum Computation- Physical Realization- Optical photon quantum computer

- By coupling this to a gate, which is opened only when a single photon (as opposed to two or more) is detected, and by appropriately delaying the outputs of multiple down-conversion sources, one can, in principle, obtain multiple single photons propagating in time synchronously, within the time resolution of the detector and gate.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Single photons can be detected with high quantum efficiency for a wide range of wavelengths, using a variety of technologies.
- For our purposes, the most important characteristic of a detector is its capability of determining, with high probability, whether zero or one photon exists in a particular spatial mode.
- For the dual-rail representation, this translates into a projective measurement in the computational basis. In practice, imperfections reduce the probability of being able to detect a single photon

Quantum Computation- Physical Realization- Optical photon quantum computer

- the *quantum efficiency* η $(0 \le \eta \le 1)$ of a photo-detector is the probability that a single photon incident on the detector generates a photo-carrier pair that contributes to detector current.

- Other important characteristics of a detector are its bandwidth (time responsivity), noise, and 'dark counts' which are photo-carriers generated even when no photons are incident.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Three of the most experimentally accessible devices for manipulating photon states are mirrors, phase shifters and beam-splitters.

- High reflectivity mirrors reflect photons and change their propagation direction in space. Mirrors with 0.01% loss are not unusual. We shall take these for granted in our scenario.

- A phase shifter is nothing more than a slab of transparent medium with index of refraction n different from that of free space, n_0; for example, ordinary borosilicate glass has $n \approx 1.5n_0$ at optical wavelengths.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Propagation in such a medium through a distance L changes a photon's phase by e^{ikL}, where $k = n\omega/c_0$, and c_0 is the speed of light in vacuum. Thus, a photon propagating through a phase shifter will experience a phase shift of $e^{i(n-n0)L\omega/c0}$ compared to a photon going the same distance through free space.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Another useful component, the beam-splitter, is nothing more than a partially silvered piece of glass, which reflects a fraction *R* of the incident light, and transmits 1−*R*. In the laboratory, a beam-splitter is usually fabricated from two prisms, with a thin metallic layer sandwiched in-between

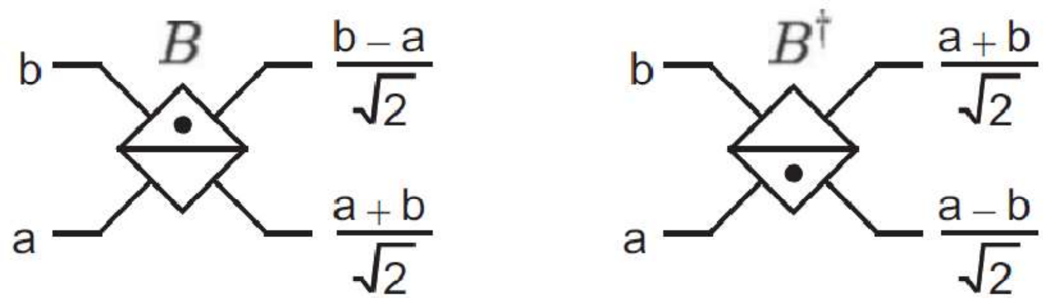
- It is convenient to define the *angle* ϑ of a beam-splitter as $\cos\vartheta = R$; note that the angle parameterizes the amount of partial reflection, and does not necessarily have anything to do with the physical orientation of the beam-splitter. The two inputs and two outputs of this device are related by

$$a_{out} = a_{in} \cos\theta + b_{in} \sin\theta$$

$$b_{out} = -a_{in} \sin\theta + b_{in} \cos\theta$$

Quantum Computation- Physical Realization- Optical photon quantum computer

- Where classically we think of *a* and *b* as being the electromagnetic fields of the radiation at the two ports.



Schematic of an optical beam-splitter, showing the two input ports, the two output ports, and the phase conventions for a 50/50 beam-splitter ($\vartheta = \pi/4$). The beam-splitter on the right is the inverse of the one on the left (the two are distinguished by the dot drawn inside). The input-output relations for the mode operators *a* and *b* are given for $\vartheta = \pi/4$.

Quantum Computation- Physical Realization- Optical photon quantum computer

- Note that in this definition we have chosen a non-standard phase convention convenient for our purposes. In the special case of a 50/50 beam-splitter, $\vartheta = 45^\circ$. Nonlinear optics provides one final useful component for this exercise: a material whose index of refraction *n* is proportional to the total intensity *I* of light going through it:

$$n(I) = n + n_2 I$$

- This is known as the optical Kerr effect, and it occurs (very weakly) in materials as mundane as glass and sugar water.

Quantum Computation- Physical Realization- Optical photon quantum computer

- In doped glasses, n_2 ranges from 10^{-14} to 10^{-7} cm^2/W, and in semiconductors, from 10^{-10} to 10^2.
- Experimentally, the relevant behavior is that when two beams of light of equal intensity are nearly co-propagated through a Kerr medium, each beam will experience an extra phase shift of $e^{in2IL\omega/c0}$ compared to what happens in the single beam case. This would be ideal if the length L could be arbitrarily long, but unfortunately that fails because most Kerr media are also highly absorptive, or scatter light out of the desired spatial mode. This is the primary reason why a single photon quantum computer is impractical