Fundamental Concepts-Introduction to Quantum Mechanics- The postulates of QM

- *All understanding begins with our not accepting the world as it appears.*

– Alan Kay

- *The most incomprehensible thing about the world is that it is comprehensible.*

– Albert Einstein

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- State space
  - The first postulate of quantum mechanics sets up the arena in which quantum mechanics takes place. The arena is our familiar friend from linear algebra, Hilbert space.

Postulate 1: Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the *state space of the* system. The system is completely described by its *state vector, which is a unit* vector in the system's state space.

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- The simplest quantum mechanical system, and the system which we will be most concerned with, is the qubit. A qubit has a two-dimensional state space. Suppose $|0\rangle$ and $|1\rangle$ form an ortho-normal basis for that state space. Then an arbitrary state vector in the state space can be written
- $|\psi\rangle = a|0\rangle + b|1\rangle$
- where a and b are complex numbers. The condition that $|\psi\rangle$ be a unit vector, $\langle\psi|\psi\rangle = 1$, is therefore equivalent to $|a|^2 + |b|^2 = 1$. The condition $\langle\psi|\psi\rangle = 1$ is often known as the normalization condition for state vectors.

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- Evolution
  - How does the state, *|ψ>, of a quantum mechanical system change with time? The following* postulate gives a prescription for the description of such state changes.

Postulate 2: The evolution of a closed quantum system is described by a unitary transformation. That is, the state |ψ> of the system at time t_1 is related to the state |ψ'> of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2 ,

- |ψ'> = U|ψ >.

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- Postulate 2 describes how the quantum states of a closed quantum system at two different times are related. A more refined version of this postulate can be given which describes the evolution of a quantum system in *continuous time.*
- **Postulate 2': The time evolution of the state of a closed quantum system is described by the *Schrödinger equation,***

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle.$$

- In this equation, is a physical constant known as *Planck's constant whose value* must be experimentally determined. The exact value is not important to us. In practice, it is common to absorb the factor into *H, effectively setting = 1. H* is a fixed Hermitian operator known as the *Hamiltonian of the closed system.*

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- Because the Hamiltonian is a Hermitian operator it has a spectral decomposition

$$H = \sum_E E|E\rangle\langle E|,$$

- with eigenvalues E and corresponding normalized eigenvectors |E>. The states |E> are conventionally referred to as energy eigenstates, or sometimes as stationary states, and E is the energy of the state |E>.

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- Quantum Measurement
  - We postulated that closed quantum systems evolve according to unitary evolution.
  - there must be times when the experimentalist and their experimental equipment observes the system to find out what is going on inside the system, an interaction which makes the system no longer closed

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- **Postulate 3: Quantum measurements are described by a collection *{Mm}* of *measurement operators*. These are operators acting on the state space of the system being measured. The index *m* refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is *|ψ>* immediately before the measurement then the probability that result *m* occurs is given by**

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

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- and the state of the system after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}$$

- The measurement operators satisfy the *completeness equation*

$$\sum_m M_m^\dagger M_m = I\,.$$

- The completeness equation expresses the fact that probabilities sum to one:
- $$1 = \sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle$$

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- A simple but important example of a measurement is the *measurement of a qubit in the computational basis*.
- This is a measurement on a single qubit with two outcomes defined by the two measurement operators $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$.
- Observe that each measurement operator is Hermitian, and that $M_0^2 = M_0$, $M_1^2 = M_1$.
- Thus the completeness relation is obeyed, $I = M_0^\dagger M_0 + M_1^\dagger M_1 = M_0 + M_1$.
- Suppose the state being measured is $|\psi\rangle = a|0\rangle + b|1\rangle$. Then the probability of obtaining measurement outcome 0 is

$$p(0) = \langle \psi | M_0^\dagger M_0 | \psi \rangle = \langle \psi | M_0 | \psi \rangle = |a|^2.$$

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- Similarly, the probability of obtaining the measurement outcome 1 is $p(1) = |b|^2$. The state after measurement in the two cases is therefore

$$\frac{M_0|\psi\rangle}{|a|} = \frac{a}{|a|}|0\rangle$$
$$\frac{M_1|\psi\rangle}{|b|} = \frac{b}{|b|}|1\rangle.$$

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- Please read the following items in the book:
  - Distinguishing quantum states
  - Projective measurements
  - POVM (Positive Operator-Valued Measure) measurements

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- Phase
  - ‘Phase’ is a commonly used term in quantum mechanics, with several different meanings dependent upon context.
  - We say that the state $e^{i\theta}|\psi\rangle$ is equal to $|\psi\rangle$, up to the global phase factor $e^{i\theta}$.
  - It is interesting to note that the *statistics of measurement predicted for these two* states are the same.
  - $\langle\psi|M^{\dagger m}M^{m}|\psi\rangle$ and $\langle\psi|e^{-i\theta}M^{\dagger m}M^{m}e^{i\theta}|\psi\rangle = \langle\psi|M^{\dagger m}M^{m}|\psi\rangle$

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- There is another kind of phase known as the *relative phase, which has quite a different* meaning. Consider the states

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

- The difference between relative phase factors and global phase factors is that for relative phase the phase factors may vary from amplitude to amplitude. This makes the relative phase a basis-dependent concept unlike global phase. As a result, states which differ only by relative phases in some basis give rise to physically observable differences in measurement statistics.

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- Composite systems
  - Suppose we are interested in a composite quantum system made up of two (or more) distinct physical systems. How should we describe states of the composite system?

Postulate 4: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

Fundamental Concepts-Introduction to Quantum Mechanics- Application: superdense coding

- *Superdense coding is a simple yet surprising application of elementary quantum mechanics.*
- An ideal example of the information processing tasks that can be accomplished using quantum mechanics
- Superdense coding involves two parties, conventionally known as 'Alice' and 'Bob', who are a long way away from one another.
- Their goal is to transmit some classical information from Alice to Bob

Fundamental Concepts-Introduction to Quantum Mechanics- Application: superdense coding

- Suppose Alice is in possession of two classical bits of information which she wishes to send Bob, but is only allowed to send a single qubit to Bob. Can she achieve her goal?
- Superdense coding tells us that the answer to this question is yes. Suppose Alice and Bob initially share a pair of qubits in the entangled state
- $$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}.$$

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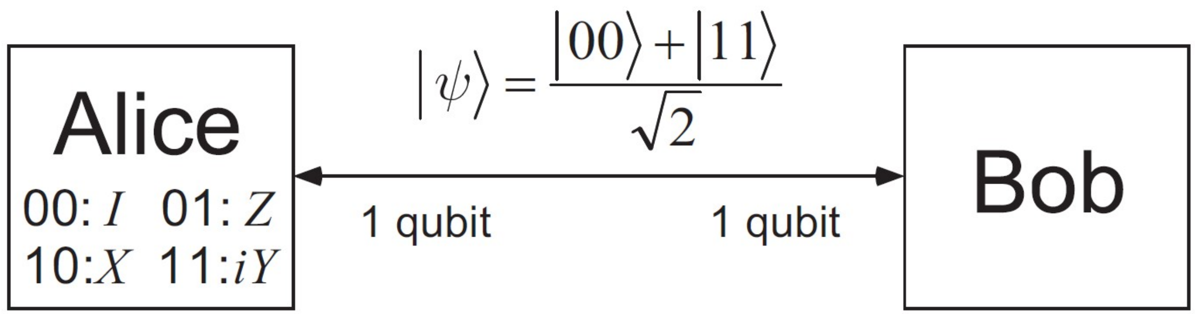
- Alice is initially in possession of the first qubit, while Bob has possession of the second qubit



- Note that |ψ> is a fixed state; there is no need for Alice to have sent Bob any qubits in order to prepare this state. Instead, some third party may prepare the entangled state ahead of time, sending one of the qubits to Alice, and the other to Bob.

Fundamental Concepts-Introduction to Quantum Mechanics- Application: superdense coding

- By sending the single qubit in her possession to Bob, it turns out that Alice can communicate two bits of classical information to Bob.
- Here is the procedure she uses:
  - If she wishes to send the bit string '00' to Bob then she does nothing at all to her qubit.
  - If she wishes to send '01' then she applies the phase flip *Z to her qubit.*
  - If she wishes to send '10' then she applies the quantum NOT gate, *X, to her qubit.*
  - If she wishes to send '11' then she applies the *iY gate to her qubit.*

Fundamental Concepts-Introduction to Quantum Mechanics- Application: superdense coding

- The four resulting states are easily seen to be:

$$
\begin{aligned}
00 &: |\psi\rangle \to \frac{|00\rangle + |11\rangle}{\sqrt{2}} \\
01 &: |\psi\rangle \to \frac{|00\rangle - |11\rangle}{\sqrt{2}} \\
10 &: |\psi\rangle \to \frac{|10\rangle + |01\rangle}{\sqrt{2}} \\
11 &: |\psi\rangle \to \frac{|01\rangle - |10\rangle}{\sqrt{2}}.
\end{aligned}
$$

- These four states are known as the *Bell basis, Bell states,* or *EPR pairs, in honor of several of the pioneers who first appreciated the novelty of* entanglement.

Fundamental Concepts-Introduction to Quantum Mechanics- Application: superdense coding

- Notice that the Bell states form an orthonormal basis, and can therefore be distinguished by an appropriate quantum measurement.
- If Alice sends her qubit to Bob, giving Bob possession of both qubits, then by doing a measurement in the Bell basis Bob can determine which of the four possible bit strings Alice sent.

Fundamental Concepts-Introduction to Quantum Mechanics- Application: superdense coding

- Summary:
  - Alice, interacting with only a single qubit, is able to transmit two bits of information to Bob.
  - Of course, two qubits are involved in the protocol, but Alice never need interact with the second qubit.
  - Classically, the task Alice accomplishes would have been impossible had she only transmitted a single classical bit, as we will show later.

Fundamental Concepts-Introduction to Quantum Mechanics- Application: superdense coding

- **Exercise: Verify that the Bell basis forms an orthonormal basis for the two qubit state space.**

- **Exercise: Suppose E is any positive operator acting on Alice's qubit. Show that $\langle\psi|E \otimes I|\psi\rangle$ takes the same value when $|\psi$ is any of the four Bell states. Suppose some malevolent third party ('Eve') intercepts Alice's qubit on the way to Bob in the superdense coding protocol. Can Eve infer anything about which of the four possible bit strings 00, 01, 10, 11 Alice is trying to send? If so, how, or if not, why not?**